Test 1 Numerical Mathematics 2
December, 2019

Duration: 1 hour.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark for this test in case of a first attempt. In case of a repair we take the minimum of the mark obtained here and a 6.

1. Consider the overdetermined problem

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$$

(a) [1.25] Use the Gram-Schmidt algorithm to make a QR factorization of the matrix.

> **Solution:** The length of the first column is 3. So the first column of Q is [1; 2 ; 2]/3. Now we have to orthogonalize the second column with respect to this column. So we want
>
> $$(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} - \frac{\alpha}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}) = 0$$
>
> which leads to $\alpha = 3$. The remaining vector is
>
> $$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$
>
> which has length $\sqrt{5}$. Hence the QR factorization is
>
> $$\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/\sqrt{5} \\ 2/3 & 1/\sqrt{5} \\ 2/3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{5} \end{bmatrix}$$

(b) [0.25] Indicate how the least-squares solution can be found using the QR-factorization.

> **Solution:** The least squares solution is found from $Rx = Q^T b$.

2. Consider $Ax = b$ with A and b given by

$$A = \begin{bmatrix} 1 & 10^{-20} \\ 2 & 2\ 10^{20} \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 2\ 10^{20} \end{bmatrix}.$$

Let the unit roundoff be given by $u = 10^{-16}$. Below you have to use this to round the intermediate results. What will be the solution if we solve the linear system, including rounding, using, Gaussian Elimination

(a) [0.5] without pivoting,

Solution: We just make the (2,1) element zero. Giving

$$A = \begin{bmatrix} 1 & 10^{-20} \\ 0 & 2\ 10^{20} - 2\ 10^{-20} \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2\ 10^{20} \quad - 2 \end{bmatrix}.$$

After dropping we obtain

$$A = \begin{bmatrix} 1 & 10^{-20} \\ 0 & 2\ 10^{20} \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2\ 10^{20} \end{bmatrix}.$$

So $x_2 = 1$ and $x_1 = 1 - 10^{-20} \approx 1$

(b) [0.75] with partial pivoting,

Solution: In the first column, the (2,1) element is bigger than the (1,1) element, so we interchange the rows and get.

$$A = \begin{bmatrix} 2 & 2\ 10^{20} \\ 1 & 10^{-20} \end{bmatrix}, \quad b = \begin{bmatrix} 2\ 10^{20} \\ 1 \end{bmatrix}.$$

Next, we make again the (2,1) element zero, giving

$$A = \begin{bmatrix} 2 & 2\ 10^{20} \\ 0 & 10^{-20} - 10^{20} \end{bmatrix}, \quad b = \begin{bmatrix} 2\ 10^{20} \\ 1 - 10^{20} \end{bmatrix}.$$

After dropping due to roundoff we obtain

$$A = \begin{bmatrix} 2 & 2\ 10^{20} \\ 0 & -10^{20} \end{bmatrix}, \quad b = \begin{bmatrix} 2\ 10^{20} \\ -10^{20} \end{bmatrix}.$$

Hence, $x_2 = 1$ and $x_1 = 2\ 10^{20} - 2\ 10^{20} = 0$

(c) [0.75] with complete pivoting,

Solution: The biggest element in the matrix is at (2,2) position, which we have to bring to (1,1) position. This gives

$$A = \begin{bmatrix} 2\ 10^{20} & 2 \\ 10^{-20} & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2\ 10^{20} \\ 1 \end{bmatrix}.$$

where now the unknown vector will be $[x_2, x_1]$ due to the column permutation. Now we make the (2,2) element zero and obtain.

$$A = \begin{bmatrix} 2\ 10^{20} & 2 \\ 0 & 1 - 10^{-40} \end{bmatrix}, \quad b = \begin{bmatrix} 2\ 10^{20} \\ 1 - 10^{-20} \end{bmatrix}.$$

After rounding we have

$$A = \begin{bmatrix} 2\ 10^{20} & 2 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2\ 10^{20} \\ 1 \end{bmatrix}.$$

which will give x_1=1 and $x_2 = 1 - 10^{-20} \approx 1$

(d) [0.75] with partial pivoting, where a row scaling is applied such that the maximum on each row of the matrix is 1.

Solution:

$$A = \begin{bmatrix} 1 & 10^{-20} \\ 10^{-20} & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Now we get

$$A = \begin{bmatrix} 1 & 10^{-20} \\ 0 & 1 - 10^{-40} \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 - 10^{-20} \end{bmatrix}.$$

After rounding we get

$$A = \begin{bmatrix} 1 & 10^{-20} \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

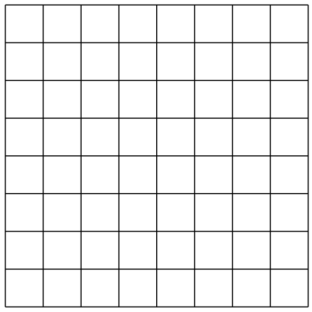
Hence $x_2 = 1$ and $x_1 = 1 - 10^{-20} \approx 1$.

(e) [0.25] Which two approaches will, in general, give the correct result?

Solution: Partial pivoting with scaling and complete pivoting give the correct result. The first one is by accident correct.

Questions continue on other side

3. Consider the graph



(a) [1.75] Copy this graph to your paper and use that to explain the Cuthill-McKee ordering on this graph. Also make a sketch of the associated vector of unknowns and the associated matrix structure such that it is clear where the unknowns and associated equations go.

Solution: Sketch attached.

(b) [0.25] Explain the relevance of reordering the matrix A, corresponding to the graph, for solving $Ax = b$

Solution: The aim of reorderings is reduce the amount of new fill during the LU factorization, and consequently also to reduce the computation time. It can be shown that there will never be fill outside the hull around the diagonal of the matrix containing all the entries. In the Cuthill-mcKee ordering one tries to make this hull as lean as possible.

4. Consider the matrix-vector multiplication $x = Ay$, where $x, y \in R^2$ and $A \in R^{2\times 2}$.

(a) [1.5] Show that in the presence of roundoff errors one is actually computing

$$\hat{x} = (A + \delta A)y$$

where $||\delta A||_\infty \leq \gamma_2 ||A||_\infty$. Here $\gamma_n = nu/(1 - nu)$ where u is the unit roundoff. You may use the lemma that $\Pi_{i=1}^n (1 + \delta_i) = 1 + \theta_n$ with $|\theta_n| \leq \gamma_n$ for $|\delta_i| < u$, $i = 1, \cdots, n$.

Solution: It holds that $x_i = a_{i1}y_1 + a_{i2}y_2$. In the computer every multiplication will produce roundoff error and therefore we have that we actually compute $\hat{x}_i = (a_{i1}y_1(1 + \delta_{i1}) + a_{i2}y_2(1 + \delta_{i2}))(1 + \delta_i)$ for some $|\delta_{ij}|, |\delta_i| < u$ hence we can replace the above line by

$$\hat{x}_i = a_{i1}y_1(1 + \theta_{2,i1}) + a_{i2}y_2(1 + \theta_{2,i2})$$

where $|\theta_{2,ij}| \leq \gamma_2$. From this, we find $\hat{x} = (A + \delta A)y$ with

$$\delta A = \begin{bmatrix} a_{11}\theta_{2,11} & a_{12}\theta_{2,12} \\ a_{21}\theta_{2,21} & a_{22}\theta_{2,22} \end{bmatrix}.$$

Using the definition of the infinity norm one can show straightforwardly that $||\delta A||_\infty \leq \max |\theta_{2,ij}| ||A||_\infty \leq \gamma_2 ||A||_\infty$.

(b) [0.5] Derive the relative condition number playing a role in the previous part

Solution: From the previous we have that $\delta x = \hat{x} - x = \delta A y$. The absolute condition number is $\max_{\delta A} ||\delta x||/||\delta A|| = \max_{\delta A} ||(\delta A)y||/||\delta A|| = ||y||$. Equality in the last step since we know for vector induced norms we have that $||(\delta A)y|| \leq ||\delta A||||y||$ moreover by chosing δA equal to identity (and also for every fraction of identity) we have equality. So we have $K_{abs} = ||y||$.
From this we easily find the relative condition number $K_{rel} = K_{abs}||A||/||x||$

(c) [0.5] Show from the previous that the matrix-vector product is forward stable.

Solution: On one hand we know that $||\delta x||/||x|| \leq K_{rel}||\delta A||/||A||$ for general perturbations δA and on the other hand that for the result with rounding it holds that $||\delta A||_\infty/||A||_\infty \leq \gamma_2$. So for the matrix vector product we have that $||\delta x||/||x|| \leq K_{rel}\gamma_2$. Which means that we can bound the relative errror in the unit-roundoff hence it is stable.